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A dispersion relation for the pion-mass dependence of hadron properties

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ABSTRACT

We present a dispersion relation in the pion-mass squared, which static quantities (nucleon mass, magnetic moment, etc.) obey under the assumption of analyticity in the entire complex m_π^2 plane modulo a cut at negative m_π^2 associated with pion production. The relation is verified here in a number of examples of nucleon and Δ -isobar properties computed in chiral perturbation theory up to order p^3 . We outline a method to obtain relations for other mass-dependencies and illustrate it on a two-loop example.

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1. Introduction

Present lattice QCD calculations are still limited to larger than physical values of light quark masses, $m_q > m_{u,d} \simeq 5\text{--}10$ MeV, but the chiral perturbation theory (χ PT) [1,2] can, in many cases, be applied to bridge the gap between the lattice and the real world (see, e.g., [3–8]). χ PT can predict at least some of the ‘non-analytic’ dependencies of static quantities (masses, magnetic moments, etc.) on pion-mass squared, or the quark mass ($m_\pi^2 \sim m_q$). The rest of the contributions contain the *a priori* unknown low-energy constants (LECs). In this Letter we examine the origins of non-analytic dependencies arising in χ PT, by considering analytic properties of the chiral expansion in the complex m_π^2 plane.

The basic observation is that chiral loops exhibit a cut along the *negative* m_π^2 axis. The cut is associated with pion production which can occur without any excess of energy for $m_\pi^2 \leq 0$. Assuming analyticity in the rest of the m_π^2 -plane (see Fig. 1), one arrives at a dispersion relation of the type:

$$f(m_\pi^2) = -\frac{1}{\pi} \int_{-\infty}^0 dt \frac{\text{Im} f(t)}{t - m_\pi^2 + i0^+}, \quad (1)$$

where f is a static quantity, 0^+ is an infinitesimally small positive number. In what follows, we explicitly verify this type of dispersion relation on a few examples of the nucleon and $\Delta(1232)$ -isobar properties and discuss its field of application. In particular, we consider a two-loop example (a sunset graph) for which the absorptive part can be extracted relatively easy. We conclude by comparing this dispersion relation with a similar ‘mass-dispersion’ relation long-known in the literature.

2. Nucleon mass

We begin right away by considering the *nucleon* properties as a function of $t = m_\pi^2$. For example, the pion-mass dependence of the nucleon mass, computed to the n th order in the chiral expansion, can be written as:

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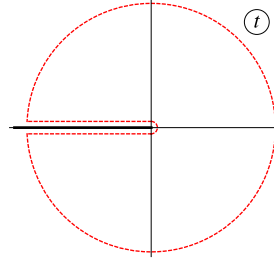


Fig. 1. The cut and the contour in the complex $t = m_\pi^2$ plane, which go into the derivation of the dispersion relation in Eq. (1).

$$M_N = \sum_{\text{even } \ell}^n a_\ell t^{\frac{\ell}{2}} + \sum_{\ell}^n \Sigma_N^{(\ell)}(t), \quad (2)$$

where a 's are some linear combinations of LECs, $\Sigma_N^{(\ell)}(t)$ is the ℓ th order nucleon self-energy given by the graphs of the type shown in Fig. 2. According to the power counting rules [9], a graph with L loops, N_π pion and N_N nucleon lines, V_k vertices from the Lagrangian of order k , contributes at order p^n , with p being the generic light scale and

$$n = \sum_k k V_k + 4L - 2N_\pi - N_N. \quad (3)$$

The leading order pion–nucleon Lagrangian is of order $k = 1$, and, to the first order in the pion-field $\pi^a(x)$ (with index $a = 1, 2, 3$), is written as [10]:

$$\mathcal{L}_{\pi N}^{(1)} = \bar{N}(x) \left(i \not{\partial} - \overset{\circ}{M}_N + \frac{\overset{\circ}{g}_A}{2 \overset{\circ}{f}_\pi} (\not{\partial} \pi^a) \tau^a \gamma_5 \right) N(x) + \text{c.t.} + \mathcal{O}(\pi^2), \quad (4)$$

where $N(x)$ is the isospin-doublet nucleon field, τ^a are Pauli matrices, $\overset{\circ}{M}_N$, $\overset{\circ}{g}_A$, and $\overset{\circ}{f}_\pi$ are respectively: the nucleon mass, axial-coupling and pion-decay constants, in the chiral limit ($m_\pi \rightarrow 0$); “c.t.” stands for counter-term contributions, which are required for the renormalization of the nucleon mass, field, and so on.

The self-energy receives its leading contribution at order p^3 , which is given by the graph Fig. 2(a) and the following expression:

$$\Sigma_N^{(3)}(t) = \frac{3g_A^2}{4f_\pi^2} i \int \frac{d^4k}{(2\pi)^4} \frac{k \cdot \gamma \gamma_5 (p \cdot \gamma - k \cdot \gamma + M_N) k \cdot \gamma \gamma_5}{(k^2 - t + i0^+)[(p-k)^2 - M_N^2 + i0^+]} \Big|_{p \cdot \gamma = M_N} \quad (5a)$$

$$\stackrel{\text{dim reg}}{=} -\frac{3g_A^2}{4f_\pi^2} \frac{M_N^3}{(4\pi)^2} \int_0^1 dx \{ [x^2 + (1-x)\tau](L_\varepsilon + \ln[x^2 + (1-x)\tau - i0^+]) + [2x^2 - (2+x)\tau](L_\varepsilon + 1 + \ln[x^2 + (1-x)\tau - i0^+]) - 3L_\varepsilon \}, \quad (5b)$$

where $\tau = t/M_N^2$, $L_\varepsilon = -1/\varepsilon - 1 + \gamma_E - \ln(4\pi \Lambda/M_N)$ exhibits the ultraviolet (UV) divergence as $\varepsilon = (4-d)/2 \rightarrow 0$, with d being the number of space–time dimensions, Λ the scale of dimensional regularization, and $\gamma_E \simeq 0.5772$ the Euler's constant. Note that for simplicity we assume the physical values for the parameters: $M_N \simeq 939$ MeV, $g_A \simeq 1.267$, $f_\pi \simeq 92.4$ MeV; the difference with the chiral-limit values leads to higher order effects.

After integration over the Feynman parameter x , this result can be written as:

$$\Sigma_N^{(3)}(t) = \frac{3g_A^2 M_N^3}{2(4\pi f_\pi)^2} \left\{ -L_\varepsilon + (1-L_\varepsilon) \frac{t}{M_N^2} \right\} + \bar{\Sigma}_N^{(3)}, \quad (6a)$$

$$\text{with } \bar{\Sigma}_N^{(3)}(t) = -\frac{3g_A^2 M_N^3}{(4\pi f_\pi)^2} \left(\tau^{3/2} \sqrt{1 - \frac{1}{4}\tau} \arccos\left(\frac{1}{2}\sqrt{\tau}\right) + \frac{1}{4}\tau^2 \ln \tau \right). \quad (6b)$$

The term in figure brackets, containing the UV-divergence, must be entirely canceled by the counter-term contribution [11], which, to this order, is of the form: $a_0 + a_2 t$, where a 's contain the “bare” values of the LECs. The first term in brackets can be viewed as a renormalization of the nucleon mass, while the second as a renormalization of the πN sigma term. The remaining part, $\bar{\Sigma}_N$, is UV-finite and consistent with the power counting in the sense that its size is indeed of order p^3 .

Let us now see whether this contribution obeys the dispersion relation of the type stated in Eq. (1). The imaginary part can be easily found from Eq. (5b) by taking into account that $\ln(-1 + i0^+) = i\pi$,

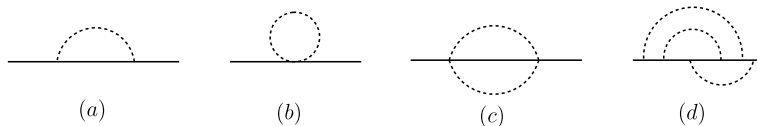


Fig. 2. Graphs representing chiral-loop corrections to the nucleon mass. Nucleon (pion) propagators are denoted by solid (dashed) lines.

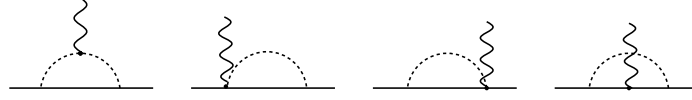


Fig. 3. Graphs representing order- p^3 corrections to the electromagnetic interaction of the nucleon.

$$\text{Im}\Sigma_N^{(3)}(t) = \frac{3g_A^2}{(4\pi f_\pi)^2} \frac{\pi}{2} \left[-(-t)^{3/2} \left(1 - \frac{t}{4M_N^2}\right)^{1/2} + \frac{t^2}{2M_N} \right] \theta(-t), \quad (7)$$

where θ is the step function. It is quite obvious that the dispersion integral with this imaginary part diverges, which is consistent with the fact that the self-energy is UV divergent. From Eq. (6a) we have seen that the divergencies appear in the first two orders of the expansion around $t = 0$ and are subsequently absorbed by the counter-terms. As the result one needs to make two subtractions at point $t = 0$ to find

$$\Sigma_N(t) - \Sigma_N(0) - \Sigma'_N(0)t = -\frac{1}{\pi} \int_{-\infty}^0 dt' \frac{\text{Im}\Sigma_N(t')}{t' - t + i0^+} \left(\frac{t}{t'}\right)^2. \quad (8)$$

Now the dispersion integral, with the imaginary part given by Eq. (7), converges, and moreover, gives the result identical to the expression in Eq. (6b), that is, the renormalized self-energy contribution. The subtractions have played here the role of the counter-terms. We therefore conclude that the order- p^3 self-energy correction to the nucleon mass obeys the suitably subtracted dispersion relation of the type of Eq. (1). We emphasize that the subtractions do not introduce any additional uncertainty in the result. The number of subtractions is not arbitrary but corresponds with the number of counter-terms available at a given order.

3. Magnetic moment and polarizability

We next turn to the example of chiral corrections to the nucleon's magnetic moment. For this we introduce the electromagnetic interaction, firstly by the minimal substitution [i.e., $\partial_\mu N \rightarrow \partial_\mu N + \frac{1}{2}(1 + \tau_3)eA_\mu N$, $\partial_\mu \pi^a \rightarrow \partial_\mu \pi^a + \epsilon^{ab3}eA_\mu \pi^b$, with $e \simeq \sqrt{4\pi/137}$] in the chiral Lagrangian, and secondly by writing out the relevant non-minimal terms:

$$\mathcal{L}_{\pi N}^{(2)} = -\frac{e}{4M_N} \bar{N} \left(\frac{1}{2}(1 + \tau_3) \overset{\circ}{\kappa}_p + \frac{1}{2}(1 - \tau_3) \overset{\circ}{\kappa}_n \right) \gamma^{\mu\nu} N F_{\mu\nu} + \text{c.t.}, \quad (9)$$

where $\overset{\circ}{\kappa}_p$ and $\overset{\circ}{\kappa}_n$ are the chiral-limit values of the proton's and neutron's anomalous magnetic moment (a.m.m.), respectively; furthermore $\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. According to the power-counting of Eq. (3), the leading order chiral correction to the electromagnetic coupling comes at order p^3 and is given by the graphs in Fig. 3. These graphs give the following contribution to the a.m.m. of, respectively, the proton and the neutron:

$$\kappa_p^{(p^3 \text{ loop})}(t) = \frac{g_A^2 M_N^2}{(4\pi f_\pi)^2} \int_0^1 dx \frac{(1 - \frac{3}{2}x)x^2}{(1-x)(t/M_N^2) + x^2 - i0^+}, \quad (10a)$$

$$\kappa_n^{(p^3 \text{ loop})}(t) = -\frac{g_A^2 M_N^2}{(4\pi f_\pi)^2} \int_0^1 dx \frac{x^2}{(1-x)(t/M_N^2) + x^2 - i0^+}. \quad (10b)$$

For negative t , these functions develop an imaginary part:

$$\text{Im}\kappa_p^{(p^3 \text{ loop})}(t) = \frac{g_A^2 M_N^2}{(4\pi f_\pi)^2} \frac{\pi}{2\lambda} \left(\frac{1}{2}\tau + \lambda \right)^2 \left(1 - \frac{3}{2} \left(\frac{1}{2}\tau + \lambda \right) \right) \theta(-t), \quad (11a)$$

$$\text{Im}\kappa_n^{(p^3 \text{ loop})}(t) = -\frac{g_A^2 M_N^2}{(4\pi f_\pi)^2} \frac{\pi}{2\lambda} \left(\frac{1}{2}\tau + \lambda \right)^2 \theta(-t), \quad (11b)$$

with $\tau = t/M_N^2$ and $\lambda = \sqrt{\frac{1}{4}\tau^2 - \tau}$. Substituting these expressions into the dispersion relation (1), and performing the integral, we obtain:

$$-\frac{1}{\pi} \int_{-\infty}^0 dt' \frac{\text{Im}\kappa_p^{(p^3 \text{ loop})}(t')}{t' - t} = \frac{g_A^2 M_N^2}{(4\pi f_\pi)^2} \frac{1}{4} \left\{ 1 - \frac{4 - 11\tau + 3\tau^2}{\sqrt{1 - \frac{1}{4}\tau}} \sqrt{\tau} \arccos \frac{\sqrt{\tau}}{2} - 6\tau + \tau(-5 + 3\tau) \ln \tau \right\}, \quad (12a)$$

$$-\frac{1}{\pi} \int_{-\infty}^0 dt' \frac{\text{Im}\kappa_n^{(p^3 \text{ loop})}(t')}{t' - t} = -\frac{g_A^2 M_N^2}{(4\pi f_\pi)^2} \frac{1}{2} \left\{ 2 - \frac{2 - \tau}{\sqrt{1 - \frac{1}{4}\tau}} \sqrt{\tau} \arccos \frac{\sqrt{\tau}}{2} - \tau \ln \tau \right\}. \quad (12b)$$

The exact same result is obtained by integrating over the Feynman parameter in the loop expressions of Eq. (10). The dispersion relation proposed in Eq. (1) is thus verified in this example as well. Note that in this case we do not need a subtraction simply because the integral converges. However, since the complete result to this order is

$$\kappa = \overset{\circ}{\kappa} + \kappa^{(p^3 \text{ loop})} + \text{c.t.}, \quad (13)$$

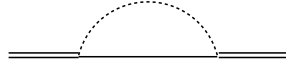


Fig. 4. A chiral loop correction to the Δ -isobar mass. Double lines denote the Δ propagation.

the counter-term contribution, which here is just a constant involving the “bare” value of a.m.m., can be put in correspondence with one subtraction at $t = 0$.

Let us remark that the same expression for the nucleon a.m.m. is obtained as well by two other dispersive methods: a derivative of the Gerasimov–Drell–Hearn sum rule [12] and a “sideways dispersion relation” [13]. Together with the present result, we therefore already have three *different* dispersion relations, which can be applied to the a.m.m. calculation. We hope that at least one of them will aid the two-loop calculation of the nucleon a.m.m.

We conclude the discussion of the nucleon properties with the example of the scalar nucleon polarizabilities: α_N (electric) and β_N (magnetic). The specifics of this example is that the leading order (p^3) correction comes entirely from chiral loops, the counter-terms are absent. In the case of magnetic polarizability of the proton, the result is given by [14,15]:

$$\beta_p^{(3)}(t) = \frac{e^2 g_A^2}{192\pi^3 f_\pi^2 M_N} \int_0^1 dx \left\{ 1 - \frac{(1-x)(1-3x)^2 + x}{(1-x)(t/M_N^2) + x^2 - i0^+} - \frac{xt/M_N^2 + x^2[1 - (1-x)(4-20x+21x^2)]}{[(1-x)(t/M_N^2) + x^2 - i0^+]^2} \right\}. \quad (14)$$

The imaginary part can be easily calculated:

$$\text{Im } \beta_p^{(3)}(t) = -\frac{e^2 g_A^2}{192\pi^3 f_\pi^2 M_N} \frac{\pi}{8\lambda^3} [2 - 72\lambda + (418\lambda - 246)\tau - (316\lambda - 471)\tau^2 + (54\lambda - 212)\tau^3 + 27\tau^4] \theta(-t), \quad (15)$$

and the dispersion relation of Eq. (1) can be shown to hold also for these expressions. The electric polarizability at order p^3 withstands this test too, however the expressions are more bulky and will be omitted here.

4. Δ -resonance

It is interesting to examine a case where the pion production cut extends into the physical region, as it happens in the case when the $\Delta(1232)$ is included as an explicit degree of freedom in the chiral Lagrangian (see, e.g., [16–20]). In this example the cut may extend from $t = -\infty$ up to $t = \Delta^2$, with $\Delta = M_\Delta - M_N$, the Delta-nucleon mass difference. The pion-mass dispersion relation for a static quantity f becomes

$$f(m_\pi^2) = -\frac{1}{\pi} \int_{-\infty}^{\Delta^2} dt' \frac{\text{Im } f(t')}{t' - m_\pi^2 + i0^+}. \quad (16)$$

Let us demonstrate how it works on the example of a chiral correction to the Δ -isobar mass. A one-loop graph with the cut all the way up to Δ^2 is shown in Fig. 4. It yields the following contribution to the self-energy [21]:

$$\Sigma_\Delta^{(\pi N \text{ loop})}(t) = -\frac{1}{2} \left(\frac{h_A M_\Delta}{8\pi f_\pi} \right)^2 \int_0^1 dx (xM_\Delta + M_N) \mathcal{M}^2(x) [L_\varepsilon + \ln \mathcal{M}^2(x)], \quad (17)$$

where $h_A \simeq 2.85$ is the $\pi N \Delta$ axial-coupling constant, and

$$\mathcal{M}^2(x) = x^2 - (1 + r^2 - \tau)x + r^2 - i0^+ = (x - \alpha)^2 - \lambda^2 - i0^+, \quad (18)$$

with $r = M_N/M_\Delta$, $\tau = t/M_\Delta^2$, $\alpha = \frac{1}{2}(1 + r^2 - \tau)$, and $\lambda^2 = \alpha^2 - r^2$.

The imaginary part arises again from the log, when its argument turns negative in the region of integration over the Feynman parameter x . More specifically,

$$\mathcal{M}^2(x) < 0, \quad \text{for } \alpha - \lambda < x < \begin{cases} 1, & \text{if } t < 0, \\ \alpha + \lambda, & \text{if } 0 \leq t \leq \Delta^2, \end{cases} \quad (19)$$

and as the result:

$$\text{Im } \Sigma_\Delta^{(\pi N \text{ loop})}(t) = \pi \frac{M_\Delta}{2} \left(\frac{h_A M_\Delta}{8\pi f_\pi} \right)^2 \times \begin{cases} \frac{1}{2}(\alpha + r)[-2\lambda^3 + (1 - \alpha)(\tau - 2\lambda^2)] + \frac{1}{4}\tau^2, & t < 0, \\ -\frac{4}{3}(\alpha + r)\lambda^3, & 0 \leq t \leq \Delta^2, \\ 0, & t > \Delta^2. \end{cases} \quad (20)$$

Note that, despite the awkward separation into regions in t , the function is continuous. For the physical value of the pion mass it provides the familiar expression for the Δ -resonance width: $\Gamma_\Delta = -2 \text{Im } \Sigma_\Delta \simeq 115 \text{ MeV}$.

We have checked that, analogously to the nucleon case, this chiral correction to the Δ mass satisfies the doubly-subtracted dispersion relation in the pion mass squared:

$$\text{Re } \Sigma_\Delta^{(\pi N \text{ loop})}(m_\pi^2) = -\frac{1}{\pi} \int_{-\infty}^{\Delta^2} dt' \frac{\text{Im } \Sigma_\Delta^{(\pi N \text{ loop})}(t')}{t' - m_\pi^2} \left(\frac{m_\pi^2}{t'} \right)^2, \quad (21)$$

where the integration is done, as usual, in the principal-value sense.

5. Direct calculation of the absorptive part

Of course to find the absorptive part it should not always be necessary to go through the entire loop calculation, as we have done so far. In the usual dispersion relations, done in external variables such as energy, the Cutkowski rules offer a simple method to compute the absorptive contributions. In our case Cutkowski rules are inapplicable because the pion is not on its positive-energy mass shell to begin with. Nevertheless a direct computation of absorptive parts is possible as will be demonstrated in the following three examples.

Consider first the tadpole, Fig. 2(b), having the following generic expression

$$J_{\text{tad}}(t) = i \int \frac{d^4 k}{(2\pi)^4} \frac{k^{2n}}{k^2 - t + i0^+}, \quad (22)$$

with an integer n . After the Wick rotation and going to the hyperspherical coordinates we obtain

$$J_{\text{tad}}(t) = \int \frac{d\Omega_4}{(2\pi)^4} \int_0^\infty dK \frac{K^{3+2n}}{K^2 + t - i0^+}, \quad (23)$$

where $\int d\Omega_4 = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta \int_0^\pi d\chi \sin^2\chi = 2\pi^2$. The absorptive part can now be simply found as

$$\text{Im } J_{\text{tad}}(t) = \frac{2\pi}{(4\pi)^2} \int_0^\infty dK K^{3+2n} \delta(K^2 + t) = \frac{\pi}{(4\pi)^2} (-t)^{1+n} \theta(-t). \quad (24)$$

As the second example we consider another typical integral, which appears, e.g., in the calculation of the graph Fig. 2(a),

$$I(t, M^2) = i \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - t + i0^+)[(p - k)^2 - M^2 + i0^+]}, \quad (25)$$

with $p^2 = M^2$. Again, after the Wick rotation and the adoption of hyperspherical coordinates we obtain

$$I(t, M^2) = - \int \frac{d\Omega_4}{(2\pi)^4} \int_0^\infty dK \frac{K^2}{(K^2 + t - i0^+)(2iM \cos\chi + K - i0^+)}, \quad (26)$$

and therefore the absorptive part is found as

$$\begin{aligned} \text{Im } I(t, M^2) &= -\pi \frac{4\pi}{(2\pi)^4} \int_0^\pi d\chi \sin^2\chi \int_0^\infty dK \frac{K^2}{2iM \cos\chi + K} \delta(K^2 + t) = -\frac{2}{(4\pi)^2} \int_0^\pi d\chi \frac{\sqrt{-t} \sin^2\chi}{2iM \cos\chi + \sqrt{-t}} \\ &= -\frac{\pi}{(4\pi)^2} \left(\frac{t}{2M^2} + \sqrt{-\frac{t}{M^2} \left(1 - \frac{t}{4M^2} \right)} \right) \theta(-t). \end{aligned} \quad (27)$$

From these two elementary examples one can see that a direct computation of the absorptive part is in principle simpler than the one of the full result, since one of the integrals is always lifted by the δ -function.

We finally come to a two-loop example, namely the pseudothreshold sunset graph with two different masses [see Fig. 2(c)]:

$$J_{\text{sunset}}(m^2, M^2) = \pi^{-d} \int d^d k_1 \int d^d k_2 \frac{1}{(k_1^2 - m^2)(k_2^2 - m^2)[(p - k_1 - k_2)^2 - M^2]}, \quad (28)$$

where $p^2 = M^2$. In this case we keep an arbitrary number of dimensions $d = 4 - 2\varepsilon$, since the absorptive part has an ultraviolet divergence. Conveniently defining the dimensionless quantities $t = m^2/M^2$ and

$$\tilde{J}(t) = \frac{M^{2(2\varepsilon-1)}}{\Gamma^2(1+\varepsilon)} J_{\text{sunset}}(m^2, M^2), \quad (29)$$

one can show that \tilde{J} satisfies a hypergeometric type of differential equation [22]:

$$t(t-1) \frac{d^2 \tilde{J}(t)}{dt^2} + \left[\frac{1}{2} - 2\varepsilon + \left(-\frac{3}{2} + 4\varepsilon \right) t \right] \frac{d\tilde{J}(t)}{dt} + \frac{1}{2} (1-2\varepsilon)(2-3\varepsilon) \tilde{J}(t) = \frac{1}{2\varepsilon^2} (t^{1-2\varepsilon} - 2t^{-\varepsilon}). \quad (30)$$

As the boundary conditions one may use the easily computable massless expressions:

$$J_{\text{sunset}}(m^2, M^2 = 0) = -m^{2(1-2\varepsilon)} \frac{\Gamma^2(1+\varepsilon)}{\varepsilon^2(1-\varepsilon)(1-2\varepsilon)}, \quad (31)$$

$$J_{\text{sunset}}(m^2 = 0, M^2) = M^{2(1-2\varepsilon)} \frac{\Gamma(3-4\varepsilon)\Gamma(2\varepsilon-1)\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)\Gamma(3-3\varepsilon)}, \quad (32)$$

hence, e.g., $t^{-1+2\varepsilon} \tilde{J}(t) \xrightarrow{t \rightarrow +\infty} -1/[\varepsilon^2(1-\varepsilon)(1-2\varepsilon)]$.

Since for real t the equation is linear with real coefficients we deduce that the solution develops an imaginary part when the inhomogeneous term (the r.h.s.) develops an imaginary part, i.e., for $t < 0$. The existence of the cut for negative m^2 is evident. Furthermore, when seeking the solution in the ε -expanded form, we again observe that solving for the imaginary part is simpler, because the corresponding inhomogeneous term is simpler at any given order in ε , cf.:

$$\frac{1}{2\varepsilon^2}t^{-\varepsilon}(t^{1-\varepsilon}-2) = \frac{\frac{1}{2}t-1}{\varepsilon^2} - \frac{t-1}{\varepsilon}\ln t + \frac{2t-1}{2}\ln^2 t + \mathcal{O}(\varepsilon), \quad (33)$$

$$\operatorname{Im} \frac{1}{2\varepsilon^2}t^{-\varepsilon}(t^{1-\varepsilon}-2) = \theta(-t)\pi \left[\frac{t-1}{\varepsilon} - (2t-1)\ln(-t) + \mathcal{O}(\varepsilon) \right]. \quad (34)$$

The solution for the imaginary part is of the form

$$\operatorname{Im} \tilde{J}(t) = \theta(-t)\pi \left[-\frac{2t}{\varepsilon} + t(-7 + (2+t)\ln(-t)) - (1-t)^2\ln(1-t) + \mathcal{O}(\varepsilon) \right], \quad (35)$$

which agrees with the result derived from a conventional formulae [23].

6. Concluding remarks

Hadron properties depend on the pion-mass squared (or, the light-quark mass, $m_{u,d} \sim m_\pi^2$) in an essentially non-analytic way. In this work we have identified the origin of this non-analyticity with a cut in the m_π^2 complex plane, which extends along the negative axis. In χ PT, the cut arises due to the possibility of a “subsoft” pion production. Assuming analyticity in the rest of the complex plane, we are able to write a simple dispersion relation in m_π^2 , cf. Eq. (1). The validity of this relation has been tested here, on a number of quantities computed in χ PT to lowest order. It also has been tested here on a generic two-loop example.

There are at least two ways in which the proposed dispersion relation can be useful. First, as a consistency constraint of various “chiral extrapolation” formulas and methods. Second, as a computational technique, similarly to how the usual dispersion relations, written in terms of energy (or, in relativistic theory, the Mandelstam variables), are used.

Although, the usual dispersion relation appears to be quite different from the dispersion relation in the mass, at a given kinematical point, the Mandelstam variables can take values given entirely in terms of mass, e.g., $t = m_\pi^2$. Then, the dispersion relation in that variable can be used to connect to another kinematical point, e.g., $t = 0$. This is precisely the strategy that had long ago been proposed to relate the scattering amplitudes at the physical pion mass with their chiral-limit values [24,25]. It led to the so-called “mass-dispersion relations” [26–28], which appear to be similar to the relation put forward in this work (cf. Eq. (2.1) of Ref. [24] with (1) here). Whether the similarities extend beyond the general form is not easy to tell since the analytic properties of the pion-mass dependence appear to be much more involved in the case of the old mass-dispersion relations. Independently of whether there is a connection between the old and new pion-mass dispersion relations, their test and validation in χ PT has been made only now.

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